5 Quadratic Functions and Equations

M11CH5

5.1. QUADRATIC FUNCTIONS

If a, b, and c are real constants with $a \neq 0$, then the function or relation of the form

$$y \equiv f(x) = ax^2 + bx + c$$

is called a **quadratic function** or **relation**. This is also known as **quadratic polynomial**.

For example, $y = x^2 - 3x - 4$, $y = 3x^2 - 5x + 2$ and $y = \sqrt{2}\ x^2 + 2.5x + 1$ are all quadratic functions.

Since, $y\ [= f(x)]$ represents a real number for any real value of x, the **domain** of a quadratic function is R, the set of all real numbers. The **range** of a quadratic function depends on the values of a, b and c.

Geometrically, the graph of any quadratic function $y = ax^2 + bx + c$, $a \neq 0$, has one of the two shapes either open upwards like $\cup$ or open downwards like $\cap$ depending on whether $a > 0$ or $a < 0$. These curves are called **parabolas**.

The zeros of a quadratic function are **those values of the variable** x **for which the function as a whole has zero value**.

For example, consider a quadratic function: $f(x) = x^2 - 3x - 4$

The zeros of the function are -1 and 4 because at $x = -1$, $f(-1) = (-1)^2 - 3(-1) - 4 = 0$ and at $x = 4$, $f(4) = (4)^2 - 3(4) - 4 = 0$.

Geometrically, for any quadratic function, the zeroes of a quadratic function $y = ax^2 + bx + c$, $a \neq 0$, are precisely the x-coordinates of the points where the parabola representing $y = ax^2 + bx + c$ intersects the x-axis. A quadratic function can have either two distinct zeroes or two equal zeroes (i.e., one zero), or no zero. This also means that a quadratic function (having degree 2) has atmost two zeroes.

5.2. QUADRATIC EQUATIONS

If a, b, and c are real numbers or constants with $a \neq 0$, then the equation of the form

$$ax^2 + bx + c = 0 \quad \ldots(1)$$

is called a **quadratic equation** in **standard form**.

For example, $2x^2 - 3x + 1 = 0$, $2x^2 + x - 300 = 0$ and $\sqrt{2}\,x^2 + 2.5x + 1 = 0$ are all quadratic equations.

In quadratic equation, highest power of variable x is two.

Clearly, if $f(x)$ is a quadratic function in x, then $f(x) = 0$ is a quadratic equation in x.

The values of x which satisfy quadratic equation (1) are called the **solutions** or **roots** or **truth values** of quadratic equation (1).

A real number α is said to *satisfy* equation (1) if $a\,\alpha^2 + b\,\alpha + c = 0$, *i.e.*, if on replacing x by α in the left hand side of equation (1), we get the right hand side zero (0).

In general, a quadratic equation has **two solutions** or **roots** or **truth values** which may be different or equal.

Every quadratic equation has exactly two real roots if $b^2 - 4ac \geq 0$. The expression $b^2 - 4ac$ is called the **discriminant** of the quadratic equation.

In general, for a quadratic equation $ax^2 + bx + c = 0$,

(*i*) If $b^2 - 4ac > 0$, then the two roots are **real** and **unequal**.

(*ii*) If $b^2 - 4ac = 0$, then the two roots are **real** and **equal**.

(*iii*) If $b^2 - 4ac < 0$, then the quadratic equation has **no real roots**.

The set of all truth values or solutions of a quadratic equation is called the **truth set** or the **solution set** of the quadratic equation. It is denoted by T or S. Thus,

$$T = \{\alpha, \beta\} \qquad \text{if } b^2 - 4ac > 0$$
$$T = \{\alpha\} \qquad \text{if } b^2 - 4ac = 0$$
$$T = \varphi \qquad \text{if } b^2 - 4ac < 0$$

A quadratic equation can be solved by a variety of methods. These are

(*i*) Factorization (*ii*) Completing the square

(*iii*) Graphical (*iv*) Quadratic formula

In the following sections, we will discuss the first three methods only assuming $b^2 - 4ac \geq 0$ so that the quadratic equation has two real roots.

5.3. SOLVING QUADRATIC EQUATIONS BY FACTORIZATION

The general steps for solving a quadratic equation by factorization are:

Step I: Write the given equation in the standard form $ax^2 + bx + c = 0$, by transposing all terms to the left hand side.

Note: If a is negative, then multiply both sides by (–1), *i.e.*, change the sign throughout so that a is positive.

Step II: Factorize the left hand side expression of the equation.

(*i*) If $b = 0$, and the quadratic in the form $ax^2 - c = 0$. We factorize LHS by using formula

$X^2 - Y^2 = (X + Y)(X - Y)$

For example: $x^2 - 9 = 0$ (Here $b = 0$), then factorize as:

$\Rightarrow x^2 - 3^2 = 0 \quad \Rightarrow \quad (x + 3)(x - 3) = 0$

(*ii*) If $c = 0$, then $ax^2 + bx = 0$. We factorize LHS by taking common from both the terms including x as:

$\Rightarrow x\,(ax + b) = 0$

For example: $2x^2 + 8x = 0$ (Here $c = 0$), then factorize as:

$\Rightarrow 2x(x + 4) = 0$

(*iii*) If a, b and c are all non-zero, then we factorize the $ax^2 + bx + c$ by finding the two numbers m and n such that $m + n = b$ (Coefficient of x) and $m \times n = ac$ (Product of coefficient of x^2 and the constant).

In the given equation, replace b by $(m + n)$, apply distributive property and group in pairs. Thus,

$$ax^2 + bx + c = 0 \quad \Rightarrow \quad (px + q)(rx + s) = 0$$

Note: Here, we get two factors $(px + q)$ and $(rx + s)$ after rearrangement of terms when we put $m + n = b$ in given equation. In the press, we split the middle term.

For example: Factorize $5x^2 - 13x - 6$. We choose two numbers whose sum is $b = -13$ and product is $ac = 5 \times (-6) = -30$, then we look for the pairs of factors of –30. The possible pairs of factors are: (30, –1), (–30, 1), (–15, 2), (15, –2), (–5, 6), (5, –6), (10, –3) and (–10, 3). But of these pairs, only the pair (2, –15) gives us –13 (i.e., the coefficient of x) when added *i.e.*, $m + n = -15 + 2 = -13$. Here, $m = -15$ and $n = 2$. Now,

$5x^2 - 13x - 6 = 5x^2 - 15x + 2x - 6$ (Replacing –13 by $m + n = -15 + 2$)

$= 5x(x - 3) + 2(x - 3)$

$= (x - 3)(5x + 2)$

Step III: Apply zero-product principle, which states that if the product of two algebraic expressions is zero, then at least one of the factors is equal to zero. [Mathematically, $ab = 0 \Rightarrow$ either $a = 0$ or $b = 0$.] Thus,

$$(px + q)(rx + s) = 0$$

$$\Rightarrow (px + q) = 0 \quad \text{or} \quad (rx + s) = 0$$

$$\Rightarrow x = -\frac{q}{p} \quad \text{or} \quad x = -\frac{s}{r}$$

Step IV: The truth set is $T = \left\{-\frac{q}{p}, -\frac{s}{r}\right\}$.

Example 1: *Find the truth sets of the following quadratic equations:*

(i) $3x^2 + 7x = 0$ (ii) $4x^2 - 49 = 0$

(iii) $2x^2 + 10 = 9x$ (iv) $9x^2 - 30x - 25 = 0$

Solution: (*i*) The given quadratic equation is:

$$3x^2 + 7x = 0$$

$$\Rightarrow x(3x + 7) = 0 \text{ (Here } c = 0)$$

By zero-product principle, we have

$$x = 0 \quad \text{or} \quad 3x + 7 = 0$$

$$\Rightarrow x = 0 \quad \text{or} \quad 3x = -7$$

$$\Rightarrow x = 0 \quad \text{or} \quad x = -\frac{7}{3}$$

Therefore, $x = 0$ and $x = -\frac{7}{3}$ are the two roots of the given equation.

Hence, the truth set $T = \left\{0, -\frac{7}{3}\right\}$

(*ii*) The given quadratic equation is:

$$4x^2 - 49 = 0$$

$\Rightarrow \quad (2x)^2 - 7^2 = 0 \quad$ (From $X^2 - Y^2$)

$\Rightarrow \quad (2x + 7)(2x - 7) = 0$

By zero-product principle, we have

$2x + 7 = 0 \quad$ or $\quad 2x - 7 = 0$

$\Rightarrow \quad 2x = -7 = 0 \quad$ or $\quad 2x = 7$

$\Rightarrow \quad x = -\frac{7}{2} \quad$ or $\quad x = \frac{7}{2}$

Therefore, $x = -\frac{7}{2}$ and $x = \frac{7}{2}$ are the two roots of the given equation.

Hence, the truth set $T = \left\{-\frac{7}{2}, \frac{7}{2}\right\}$

(*iii*) The given quadratic equation is:

$2x^2 + 10 = 9x$

Tran sposing all terms to left hand side,

$2x^2 - 9x + 10 = 0$

Here, $a = 2$, $b = -9$, $c = 10$

The two numbers whose sum is $b = -9$ and product is $ac = 2 \times 10 = 20$ are -5 and -4.

Replacing -9 by $-5 - 4$, we get

$2x^2 - 5x - 4x + 10 = 0$

$\Rightarrow \quad x(2x - 5) - 2(2x - 5) = 0$

$\Rightarrow \quad (2x - 5)(x - 2) = 0$

By zero-product principle, we have

$2x - 5 = 0 \quad$ or $\quad x - 2 = 0$

$\Rightarrow \quad 2x = 5 \quad$ or $\quad x = 2$

$\Rightarrow \quad x = \frac{5}{2} \quad$ or $\quad x = 2$

Therefore, $x = \frac{5}{2}$ and $x = 2$ are the two roots of the given equation.

Hence, the truth set $T = \left\{\frac{5}{2}, 2\right\}$.

(*iv*) The given quadratic equation is

$$9x^2 - 30x + 25 = 0$$

$$\Rightarrow \quad (3x)^2 - 2(3x)(5) + 5^2 = 0$$

$$\Rightarrow \quad (3x - 5)^2 = 0$$

$$\Rightarrow \quad (3x - 5)(3x - 5) = 0$$

By zero-product principle, we have

$$3x - 5 = 0 \quad \text{or} \quad 3x - 5 = 0$$

$$\Rightarrow \quad 3x = 5 \quad \text{or} \quad 3x = 5$$

$$\Rightarrow \quad x = \frac{5}{3} \quad \text{or} \quad x = \frac{5}{3}$$

Therefore, $x = \frac{5}{3}$ and $x = \frac{5}{3}$ are the two roots of the given equation.

Hence, the truth set $T = \left\{\frac{5}{3}\right\}$.

Example 2: *Solve the following quadratic equations:*

(*i*) $-2x^2 + 3x - 1 = 0$ (*ii*) $12x^2 - 7x + 1 = 0$

(*iii*) $2x^2 + 7x + 3 = 0$ (*iv*) $6x^2 - x - 2 = 0$

Solution:

(*i*) The given quadratic equation is:

$$-2x^2 + 3x - 1 = 0$$

$$2x^2 - 3x + 1 = 0 \qquad \text{(Multiplying by –1 both sides)}$$

$$\Rightarrow \quad 2x^2 - 2x - x + 1 = 0 \qquad \text{(Splitting the middle term)}$$

$$\Rightarrow \quad 2x(x - 1) - 1(x - 1) = 0 \qquad \text{(Common factor from the two pairs)}$$

$$\Rightarrow \quad (x - 1)(2x - 1) = 0 \qquad \text{(Factor form)}$$

$$\Rightarrow \quad x - 1 = 0 \quad \text{or} \quad 2x - 1 = 0 \qquad \text{(By zero-product principle)}$$

$$\Rightarrow \quad x = 1 \quad \text{or} \quad x = \frac{1}{2}$$

$$\Rightarrow \quad x = 1 \quad \text{or} \quad \frac{1}{2}$$

(*ii*) The given quadratic equation is:

$$12x^2 - 7x + 1 = 0$$

$\Rightarrow$ $12x^2 - 4x - 3x + 1 = 0$ (Splitting the middle term)

$\Rightarrow$ $4x(3x - 1) - 1(3x - 1) = 0$ (Common factor from the two pairs)

$\Rightarrow$ $(3x - 1)(4x - 1) = 0$ (Factor form)

$\Rightarrow$ $3x - 1 = 0$ or $4x - 1 = 0$ (By zero-product principle)

$\Rightarrow$ $x = \frac{1}{3}$ or $x = \frac{1}{4}$

$\Rightarrow$ $x = \frac{1}{3}$ or $\frac{1}{4}$

(*iii*) The given quadratic equation is:

$$2x^2 + 7x + 3 = 0$$

$\Rightarrow$ $2x^2 + 6x + x + 3 = 0$ (Splitting the middle term)

$\Rightarrow$ $2x(x + 3) + 1(x + 3) = 0$ (Common factor from the two pairs)

$\Rightarrow$ $(x + 3)(2x + 1) = 0$ (Factor form)

$\Rightarrow$ $x + 3 = 0$ or $2x + 1 = 0$ (By zero-product principle)

$\Rightarrow$ $x = -3$ or $x = -\frac{1}{2}$

$\Rightarrow$ $x = -3$ or $-\frac{1}{2}$

(*iv*) The given quadratic equation is:

$$6x^2 - x - 2 = 0$$

$\Rightarrow$ $6x^2 - 4x + 3x - 2 = 0$ (Splitting the middle term)

$\Rightarrow$ $2x(3x - 2) + 1(3x - 2) = 0$ (Common factor from the two pairs)

$\Rightarrow$ $(3x - 2)(2x + 1) = 0$ (Factor form)

$\Rightarrow$ $3x - 2 = 0$ or– $2x + 1 = 0$ (By zero-product principle)

$\Rightarrow$ $x = \frac{2}{3}$ or $x = -\frac{1}{2}$

$\Rightarrow$ $x = \frac{2}{3}$ or $-\frac{1}{2}$

Example 3: *Find the roots of the following quadratic equations by factorization:*

(*i*) $x^2 - 22x + 120 = 0$ (*ii*) $3x^2 - x - 4 = 0$

(*iii*) $6x^2 + 5x - 6 = 0$ (*iv*) $3x^2 - 2\sqrt{6}\ x + 2 = 0$

Solution: (*i*) The given quadratic equation is:

$$x^2 - 22x + 120 = 0$$

$$x^2 - 12x - 10x + 120 = 0 \quad \text{(Splitting the middle term)}$$

$$\Rightarrow x(x - 12) - 10(x - 12) = 0 \quad \text{(Common factor from the two pairs)}$$

$$\Rightarrow (x - 12)(x - 10) = 0 \quad \text{(Factor form)}$$

$$\Rightarrow x - 12 = 0 \text{ or } x - 10 = 0 \quad \text{(By zero-product principle)}$$

$$\Rightarrow x = 12 \text{ or } x = 10$$

Hence, the roots of the given equations are 10 and 12.

(*ii*) The given quadratic equation is:

$$3x^2 - x - 4 = 0$$

$$\Rightarrow 3x^2 + 3x - 4x - 4 = 0 \quad \text{(Splitting the middle term)}$$

$$\Rightarrow 3x(x + 1) - 4(x + 1) = 0 \quad \text{(Common factor from the two pairs)}$$

$$\Rightarrow (x + 1)(3x - 4) = 0 \quad \text{(Factor form)}$$

$$\Rightarrow x + 1 = 0 \text{ or } 3x - 4 = 0 \quad \text{(By zero-product principle)}$$

$$\Rightarrow x = -1 \text{ or } x = \frac{4}{3}$$

Hence, the roots of the given equation are -1 and $\frac{4}{3}$.

(*iii*) The given quadratic equation is:

$$6x^2 + 5x - 6 = 0$$

$$\Rightarrow 6x^2 + 9x - 4x - 6 = 0 \quad \text{(Splitting the middle term)}$$

$$\Rightarrow 3x(2x + 3) - 2(2x + 3) = 0 \quad \text{(Common factor from the two pairs)}$$

$$\Rightarrow (2x + 3)(3x - 2) = 0 \quad \text{(Factor form)}$$

$$\Rightarrow 2x + 3 = 0 \text{ or } 3x - 2 = 0 \quad \text{(By zero-product principle)}$$

$$\Rightarrow x = \frac{3}{2} \text{ or } x = \frac{2}{3}$$

Hence, the roots of the given equation are $\frac{2}{3}$ and $-\frac{3}{2}$.

(*iv*) The given quadratic equation is:

$$3x^2 - 2\sqrt{6}\,x + 2 = 0$$

$\Rightarrow 3x^2 - \sqrt{6}\,x - \sqrt{6}\,x + 2 = 0$ (Splitting the middle term)

$\Rightarrow (\sqrt{3}\,x)^2 - \sqrt{6}\,x - \sqrt{6}\,x + (\sqrt{2})^2 = 0$

$\Rightarrow \sqrt{3}\,x(\sqrt{3}\,x - \sqrt{2}) - \sqrt{2}\,(\sqrt{3}\,x - \sqrt{2}) = 0$ (Common factor from the two pairs)

$\Rightarrow (\sqrt{3}\,x - \sqrt{2})\,(\sqrt{3}\,x - \sqrt{2}) = 0$ (Factor form)

$\Rightarrow \quad (\sqrt{3}\,x - \sqrt{2}) = 0$ or $(\sqrt{3}\,x - \sqrt{2}) = 0$ (By zero-product principle)

$\Rightarrow \quad x = \dfrac{\sqrt{2}}{\sqrt{3}}$ or $x = \dfrac{\sqrt{2}}{\sqrt{3}}$

So, roots are equal. Hence, the roots of the given equation are $\dfrac{\sqrt{2}}{\sqrt{3}}$ and

Forming the Quadratic Equation when Two Roots are Given

In general, if α and β are the given roots of any quadratic equation, then we can write the quadratic equations as:

$$x^2 - (\alpha + \beta)\,x + \alpha\beta = 0$$

$\Rightarrow \quad x^2 -$ (**Sum of roots**) $x +$ (**Product of roots**) $= 0$

Example 4: *Find the quadratic equations having the following roots:*

(*i*) *2 and 3* (*ii*) *3 and –5* (*iii*) *–5 and –12* (*iv*) $\dfrac{1}{3}$ *and* $-\dfrac{2}{3}$

Solution: (*i*) Let α and β be the roots of the given quadratic equation. Then

Sum of roots = $\alpha + \beta = 2 + 3 = 5$

Product of roots = $\alpha\beta = 2 \times 3 = 6$

The required quadratic equation is given by:

$x^2 -$ (Sum of roots) $x +$ (Product of roots) $= 0$

$\Rightarrow \quad x^2 - 5x + 6 = 0$

(*ii*) Let α and β be the roots of the given quadratic equation. Then

Sum of roots = $\alpha + \beta = 3 + (-5) = -2$

Product of roots = $\alpha\beta = 3 \times (-5) = -15$

The required quadratic equation is given by:

$\Rightarrow \quad x^2 - (\text{Sum of roots})\ x + (\text{Product of roots}) = 0$

$\Rightarrow \quad x^2 - (-2)x - 15 = 0$

$\Rightarrow \quad x^2 + 2x - 15 = 0$

(*iii*) Let α and β be the roots of the given quadratic equation. Then

Sum of roots = $\alpha + \beta = -5 + (-12) = -17$

Product of roots = $\alpha\beta = (-5) \times (-12) = 60$

The required quadratic equation is given by:

$x^2 - (\text{Sum of roots})\ x + (\text{Product of roots}) = 0$

$\Rightarrow \quad x^2 - (-17)x + 60 = 0$

$\Rightarrow \quad x^2 + 17x + 60 = 0$

(*iv*) Let α and β be the roots of the given quadratic equation. Then

Sum of roots = $\alpha + \beta = \frac{1}{3} + \left(-\frac{2}{3}\right) = \frac{1}{3} - \frac{2}{3} = -\frac{1}{3}$

Product of roots = $\alpha\beta = \frac{1}{3} \times \left(-\frac{2}{3}\right) = -\frac{2}{9}$

The required quadratic equation is given by:

$x^2 - (\text{Sum of roots})\ x + (\text{Product of roots}) = 0$

$$x^2 - \left(-\frac{1}{3}\right)x - \frac{2}{9} = 0$$

$$\Rightarrow \quad x^2 + \frac{1}{3}x - \frac{2}{9} = 0$$

$$\Rightarrow \quad 9x^2 + 3x - 2 = 0$$

EXERCISE 5.1

1. Find the truth sets of the following quadratic equations by factoriation:

(i) $x^2 - 4 = 0$ (ii) $x^2 - 16 = 0$

(iii) $x^2 - 5x = 0$ (iv) $3x^2 - 21x = 0$

(v) $x^2 - 2x - 3 = 0$ (vi) $x^2 + 2x = 35$

(vii) $3x^2 - 8x + 5 = 0$ (viii) $x^2 = 3(2x + 9)$

(ix) $2x^2 - x = 3$ (x) $6x^2 - 23x - 55 = 0$

2. Solve the following quadratic equations by factorisation:

(*i*) $2x^2 - 5x = 0$ (*ii*) $5x^2 = 8x$

(*iii*) $5x^2 = 20$ (*iv*) $4x^2 - 7 = 0$

(*v*) $x^2 - 3x - 4 = 0$ (*vi*) $x^2 - 4x + 3 = 0$

(*vii*) $x^2 + 7x + 10 = 0$ (*viii*) $3x^2 + 5x - 2 = 0$

3. Find the roots of the following quadratic equations by factorisation:

(*i*) $x^2 + 9x - 52 = 0$ (*ii*) $2x^2 + 5x - 12 = 0$

(*iii*) $2x^2 + 2 = 5x$ (*iv*) $6x^2 + 5x = 4$

(*v*) $3x^2 + 14x + 8 = 0$ (*vi*) $7x^2 = 8 - 10x$

(*vii*) $4x^2 + 12x + 9 = 0$ (*viii*) $(x - 1)^2 = 4$

4. Find the quadratic equations having the following roots:

(*i*) 1 and 3 (*ii*) 4 and – 9

(*iii*) – 4 and 3 (*iv*) 1 and $\frac{3}{2}$

(*v*) 3 and $\frac{2}{3}$ (*vi*) $-\frac{1}{2}$ and 2

(*vii*) $\frac{1}{2}$ and $\frac{3}{4}$ (*viii*) $-\frac{1}{4}$ and $-\frac{2}{3}$

5.4. WORD–PROBLEMS BASED ON QUADRATIC EQUATIONS

In this Section, we shall discuss some applications of quadratic equations in day-to-day life situations. There are several situations in the world around us and in different fields of Mathematics wherein quadratic equations arise. In such situations, the general technique is to first formulate a quadratic equation by introducing the unknown quantity as the variable x and then to find the solution of the resulting quadratic equation in x.

4-Step Working Rule for Solution of Word Problems

We suggest the following 4-*Step Working Rule (WR)* to solve word problems based on quadratic equations:

Step 1 *First of all, formulate a quadratic equation representing the given problem byintroducing the variable x.*

Step 2 *Solve the quadratic equation (or system of equations) as obtained in Step 1, by using the quadratic formula orFactorization method.*

Step 3 *Out of two roots of the quadratic equation select the one which satisfies the condition stipulated in the given problem and reject the one which does not satisfy the problem.*

Step 4 *Finally, translate the required solution of the quadratic into language of the problem.The technique of above WR is further eleborated below by taking several problems of various types from daily life situations.*

Example 5: *Represent the following situation in the form of a quadratic equations. The product of two consecutive positive integers is 306. We need to find the integers.*

Solution: Let the two consecutive positive integers by, x and $x + 1$.

As per given,

Product of two integers = 306

$\Rightarrow \quad (x) \times (x + 1) = 306 \quad \Rightarrow \quad x^2 + x = 306$

$\Rightarrow \quad x^2 + x = 306 = 0$ ...(1)

which is the required quadratic equation. Solve it by factarization as:

$x^2 + x - 306 = 0$

$\Rightarrow \quad x^2 + 18x - 17x - 306 = 0$ [Splitting the middle term]

$\Rightarrow \quad x\,(x + 18) - 17\,(x + 18) = 0$

$\Rightarrow \quad (x + 18)\,(x - 17) = 0$

$\Rightarrow \quad x + 18 = 0$ or $x = -17$

$\Rightarrow \quad x = -18$ or $x = 17$

$\Rightarrow \quad x = 17, -18$ [Rejecting -18, $\because$ it is not a positive integer]

Therefore, two numbers are:

1st positive integer = 17

2nd consecutive positive integer = 17 + 1 = 18

Hence, the required two consecutive positive integers are 17 and 18.

Example 6: *Find the two numbers whose sum is 27, and product is 182.*

Solution: Let the first number be x and the second number by y.

As per given,

$$x + y = 27 \quad ...(1) \quad \text{and} \quad x \times y = 182 \quad ...(2)$$

From (1), $x = 27 - y$

Putting $x = 27 - y$ in (2), we get

$$(27 - y)\, y = 182$$

$$\Rightarrow \quad 27 - y^2 = 182$$

$$\Rightarrow \quad y^2 - 27y + 182 = 0 \quad ...(3)$$

which is a quadratic equation in y.

$y^2 - 27y + 182 = 0 \Rightarrow y^2 - 14y + 13y + 182 = 0$

[Splitting the middle term]

$y(y - 14) - 13\,(y - 14) = 0 \quad \Rightarrow \quad (y - 14)(y - 13) = 0$

$\Rightarrow \quad y - 14 = 0 \quad$ or $\quad y - 13 = 0$

$\Rightarrow \quad y = 14 \quad$ or $\quad y = 13$

$\Rightarrow \quad y = 13, 14$

Putting the value of y in (1), we get

$x + 14 = 27$	$x + 13 = 27$
$x = 27 - 14$	$x = 27 - 13$
$x = 13$	$x = 14$

Hence, the required two numbers are 13 and 14.

Example 7: *Find two consecutive positive integers, sum of whose squares is 365.*

Solution:

Let two consecutive integers be x and $x + 1$.

As per given,

$$(x^2) + (x + 1)^2 = 365$$

$$\Rightarrow \quad x^2 + x^2 + 1 + 2x = 365 \qquad \because (a + b)^2 = a^2 + b^2 + 2ab$$

$\Rightarrow \quad 2x^2 + 2x - 364 = 0 \quad \Rightarrow \quad 2(x^2 + x - 182) = 0$

$\Rightarrow \quad x^2 - x - 182 = 0$ ...(1)

which is a quadratic equation in x.

$x^2 + x - 182 = 0$

$\Rightarrow \quad x^2 + 14x - 13x - 182 = 0$ [Splitting the middle term]

$\Rightarrow \quad x^2 + 14x - 13x - 182 = 0$

$\Rightarrow \quad x(x + 14) - 13(x + 14) = 0 \quad \Rightarrow \quad (x - 13)(x + 13) = 0$ ...(2)

$\Rightarrow \quad x - 13 = 0$ or $x + 14 = 0$

$\Rightarrow \quad x = 13$ or $x = -14$

$\Rightarrow \quad x = 13, -14$ (Rejecting –14 ∵ it is not a positive integer)

$\Rightarrow \quad x = 13$

1st positive integer = 13

2nd consecutive positive integer = 13 + 1 = 14

Hence, the required two consecutive positive integers are 13 and 14.

Example 8: *Represent the following situtation in the form of a quadratic equation. Albart mother is 26 years older than him. The product of their ages (in years) 3 years from now will be 360. We would like to find Albart present age.*

Solution:

Suppose Albart present age = x years

∴ Albart mother's present age = $(26 + x)$ years

After 3 years Albart age = $x + 3$

His mother's age = $(26 + x) + 3 = 29 + x$

According to the given problem,

$(29 + x)(x + 3) = 360$

$\Rightarrow \quad 29x + 87 + x^2 + 3x = 360$

$\Rightarrow \quad x^2 + 32x - 273 = 0$

which is the required quadratic equation.

Now, $x^2 + 32x - 273 = 0 \quad \Rightarrow \quad x^2 + 39x - 7x - 273 = 0$

$\Rightarrow \quad x(x + 39) - 7(x + 39) = 0 \quad \Rightarrow \quad (x + 39)(x - 7) = 0$

$\Rightarrow \quad x + 39 = 0$ or $x - 7 = 0$

$\Rightarrow \quad x = -39$ or $x = 7$

$\Rightarrow \quad x = -39, 7$

$\Rightarrow \quad x = 7$ (Rejecting $x = -39$ $\because$ age cannot be negative)

Hence, the present age of Albart is 7 years.

Example 9: *Represent the following situation in the form of a quadratic equation The area of a rectangular plot is 528 m*^{2}*. The length of the plot (in metres) is one more than twice its breadth. We need to find the lengfh and breadth of the plot.*

Solution:

Suppose, breadth of the rectangular plot, $b = x\ m$

Then, length of the rectangular plot, $l = (2x + 1)$ m

As per given,

Length × Breadth = Area of rectangular plot

$\Rightarrow \quad l \times b = 528$

$\Rightarrow \quad (2x + 1)\ x = 528$

$\Rightarrow \quad 2x^2 + x = x = 528$

$\therefore \quad 2x^2 + x - 528 = 0$ ...(1)

[which is the required quadratic equation]

Now, $2x^2 + x - 528 = 0 \Rightarrow 2x^2 + 33x - 32x - 32x - 528 = 0$

$\Rightarrow \quad x(2x + 33) - 16(2x + 33) = 0 \Rightarrow (2x + 33)(x - 16)$

$\Rightarrow \quad 2x + 33 = 0$ or $x - 16 = 0$

$\Rightarrow \quad x = -\frac{33}{2}$ or $x = 16$

$\Rightarrow \quad x = -\frac{32}{3} 16,$

$\Rightarrow \quad x = 16$ (Rejecting $x = -\frac{33}{2}$, since length can't be negative)

Hence, breadth of the plot = 16 m and length of the plot = 2 16 + 1 = 33 m,

EXERCISE 5.2

1. The product of two consecutive odd positve numbers is 195. By constructing a quadratic equation and solving it, find the two numbers.

2. The sum of the squares of two consecutive natural numbers is 313. Find the numbers.

3. Find two consecutive odd positive integers, sum of whose squares is 290.

4. Divide 16 into two parts such that twice the square of the larger part exceeds the square of the smaller part by 164.

5. The sum of the squares of two positive integers is 208. If the square of the larger number is 18 times the smaller number, find the numbers.

6. The difference of the squares of two numbers is 45. The square of the smaller number is 4 times the larger number. Determine the numbers.

7. One year ago, a man was 8 times as old as his son. Now his age is equal to the square of his son's age. Find their persent ages.

8. The product of Ramu's age (in years) five years ago with his age (in years) 9 years later is 15. Find Ramu's present age.

9. The sum of ages of a father and his son is 45 years. Five years ago, the product of their ages (in years) was 124. Determine their present ages.

10. Seven years ago, let Swati's age be x years. Then, seven years ago Varun's ago was $5x^2$ years.

11. The hypotenuse of right-angled triangle is 6 metres more than twice the shortest side. If the third side is 2 meters less than the hypotenuse, find the side of the triangle.

12. The hypotenuse of a grassy land in the shape of a right triangle is 1 metre more than twice the shortest side. If the third side is 7 metres more than the shortest side, find the side of the grassy land.

13. The hypotenuse of a right triangle is $3\sqrt{5}$ cm. If the smaller side is tripled and the larger side is doubled, the new hypotenuse will be 15 cm. Find the length of each side.

14. Vikram wishes to fit three rods together in the shape of a right triangle. The hypotenuse is to be 2 cm longer than the base and 4 cm longer than the altitude. What should be the lengths of the rods?

15. The length of a rectangle exceeds its width by 8 cm and the area of the rectangle is 240 sq. cm. Find the dimensions of the rectangle.

16. The side of a square exceeds the side of the another square by 4 cm and the sum of the areas of the two squares is 400 sq. cm. Find the dimensions of the squares.

5.5. QUADRATIC EQUATIONS BY COMPLETING THE SQUARES (OPTIONAL)

In this Section, we shall study another method of solving a quadratic equation which is called the Method of Completing the Squares. The key-point involved in this method is making a perfect square of the quadratic trinomial by adding and subtracting the square of half of the coefficient of x throughout, with the leading coefficient unity (*i.e.*, 1).

Note: Solving an equation by the Method of Completing the Square is also known as **Solving by first principles** (*or* **5-Step Method**).

Steps for Completing the square method:

Suppose $ax^2 + bx + c = 0$ is the given quadratic equation. Then follow the given steps to solve it by completing the square method.

Step 1: If a is not equal to 1, then divide the complete equation by a such that the cieffiicient of x^2 will be 1. For example,

$$x^2 + \left(\frac{b}{a}\right)x + \left(\frac{c}{a}\right) = 0$$

Step 2: Add the square of half of the coefficient of term $x, \left(\frac{b}{2a}\right)^2$, on both sides. For example,

$$x^2 + \left(\frac{b}{a}\right)x + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 + \left(\frac{c}{a}\right) = 0$$

Step 3: Factorize the left hand side (LHS) of the equation as the square of the binomial term and then take constant terms on right hand side (RHS). For example,

$$\left[x+\left(\frac{b}{2a}\right)\right]^2-\left(\frac{b}{2a}\right)^2+\left(\frac{c}{a}\right)=0$$

$$\Rightarrow \quad \left[x+\left(\frac{b}{2a}\right)\right]^2-\left[\frac{b^2}{4a^2}+\frac{c}{a}\right]=0$$

$$\Rightarrow \quad \left[x+\left(\frac{b}{2a}\right)\right]^2-\left[\frac{b^2-4ac}{4a^2}\right]=0$$

$$\Rightarrow \quad \left[x+\left(\frac{b}{2a}\right)\right]^2=\left[\frac{b^2-4ac}{4a^2}\right]$$

Step 4: Take the square root on both the side.

For example, if $b^2 - 4ac \geq 0$, then by taking the square root, we get

$$x+\left(\frac{b}{2a}\right)=\pm\sqrt{\frac{b^2-4ac}{4a^2}}$$

Step 5: Solve for variable x and find the roots after further simplification. For example,

$$\Rightarrow x=\pm\frac{\sqrt{b^2-4ac}}{2a}-\frac{b}{2a}$$

$$\Rightarrow x+\left(\frac{b}{2a}\right)=\pm\frac{\sqrt{b^2-4ac}}{2a}$$

This is the required solution or roots of the given quadratic.

The procedure for the above method is exhibited below in the following examples for a better understanding for this technique.

Example 10: *Find the roots of the following quadratic equations (if they exist) by the method of completing the squares:*

(*i*) $2x^2 - 7x + 3 = 0$ (*ii*) $2x^2 + x + 4 = 0$

(*iii*) $2x^2 + 8x - 10 = 0$

Solution:

(*i*) The given quadratic equation is:

$$2x^2 - 7x + 3 = 0$$

Dividing throughout by 2 (*i.e.*, the coefficient of x^2), we get

$$x^2 - \frac{7}{2}x + \frac{3}{2} = 0$$

Adding to both sides, $\left(\frac{1}{2}\times\frac{7}{2}\right)^2$ (*i.e.*, square of $\frac{1}{2}$ the coefficient of x), we have

$$x^2 - 2\left(\frac{1}{2}\times\frac{7}{2}\right)(x) + \left(\frac{1}{2}\times\frac{7}{2}\right)^2 - \left(\frac{1}{2}\times\frac{7}{2}\right)^2 + \frac{3}{2} = 0$$

$$\Rightarrow \quad x^2 - 2(x)\left(\frac{7}{4}\right) + \left(\frac{7}{4}\right)^2 - \frac{49}{16} + \frac{3}{2} = 0$$

$$\Rightarrow \quad \left(x - \frac{7}{4}\right)^2 - \frac{49-(3)(8)}{16} = 0 \quad \left(x - \frac{7}{4}\right)^2 = \frac{49-24}{16} = \frac{25}{16}$$

Taking square root of both sides, we get

$$\left(x - \frac{7}{4}\right) = \pm\sqrt{\frac{25}{16}} = \pm\frac{5}{4}$$ [Here, "±" ⇒ plus and minus]

Again, transposing,

$$x = \frac{7}{4} \pm \frac{5}{4}$$

$$\Rightarrow x = \frac{7}{4} + \frac{5}{4} = \frac{12}{4} = 3 \quad \text{or} \quad x = \frac{7}{4} - \frac{5}{4} = \frac{2}{4} = \frac{1}{2}$$

Hence, the roots of the given equation are 3 and $\frac{1}{2}$.

(*ii*) The given quadratic equation is:

$$2x^2 + x + 4 = 0$$

Dividing throughout by 2 (*i.e.*, the coefficient of x^2), we get

$$x + \frac{x}{2} + \frac{4}{2} = 0 \quad \Rightarrow \quad x^2 + \frac{x}{2} + 2 = 0$$

Adding to both sides, $\left[\frac{1}{2}\times\frac{1}{2}\right]^2$ (*i.e.*, square of $\frac{1}{2}$ the coefficient of x), we have

$$x^2 + 2\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)x + \left(\frac{1}{4}\right)^2 - \left(\frac{1}{4}\right)^2 + 2 = 0$$

$$\Rightarrow \quad \left(x + \frac{1}{4}\right)^2 - \frac{1}{16} + 2 = 0$$

$$\Rightarrow \quad \left(x + \frac{1}{4}\right)^2 - \frac{1 - 2(16)}{16} = 0$$

$$\Rightarrow \quad \left(x + \frac{1}{4}\right)^2 + \frac{31}{16} = 0$$

$$\Rightarrow \quad \left(x + \frac{1}{4}\right)^2 = \frac{-31}{16} < 0$$

which is not possible, since $\left(x + \frac{1}{4}\right)^2$ cannot be negative for any real value of x.

So, there is no real value of x satisfying the given equation. That is the given equation has no real roots.

Remark: The above method is, no doubt, elaborate and lengthy. It should be followed only when specifically asked in a problem.

(*iii*) The given quadratic equation is:

$2x^2 + 8x - 10 = 0$

Dividing throughout by 2 (*i.e.*, the coefficient of x^2), we get

$$x^2 + 4x - 5 = 0$$

Adding to both sides, $\left[\frac{1}{2} \times 4\right]^2$

(*i.e.*, square of $\frac{1}{2}$ the coefficient of x), we have

$$x^2 + 2\left(\frac{1}{2}\right)4x + 2^2 - 2^2 - 5 = 0$$

$$\Rightarrow \quad (x + 2)^2 - 4 - 5 = 0$$

$$\Rightarrow \quad (x + 2)^2 - (4 + 5) = 0$$

$$\Rightarrow \quad (x + 2)^2 = 9 > 0$$

Taking square root of both sides, we get

$$x^2 + 2 = \pm\sqrt{9} = \pm 3$$

Again transposing,

$x = \pm 3 - 2$

$\Rightarrow$ $x = 0, -5$

Therefore, the roots of the given equation are 1 and –5.

EXERCISE 5.3

Find the roots of the following quadratic equations (if they exist) by the method of completing the squares.

1. $9x^2 - 15x + 6 = 0$
2. $2x^2 - 5x + 3 = 0$
3. $5x^2 - 6x - 2 = 0$
4. $4x^2 + 3x + 5 = 0$

5.6. GRAPHS OF QUADRATIC FUNCTIONS

A function of the form $f(x) = ax^2 + bx + c$, where a, b and c are real numbers and a ≠ 0 is called a **quadratic function**. The graph of a quadratic function is a curve called **parabola**. Parabolas are shaped like cups or caps (Figs. (*i*) and (*ii*)).

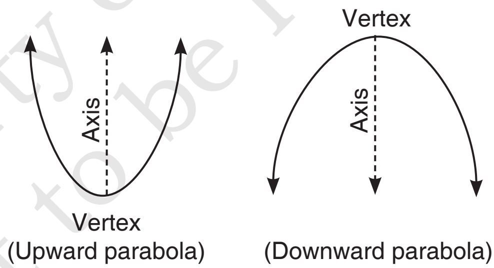

Fig. 5.1.

If a, the coefficient of x^2, is positive, the parabola is like a cup and opens upwards.

If a, the coefficient of x^2, is negative, the parabola is like a cap and opens downwards.

There is a turning point on either parabola. This turning point is called the **vertex** of the parabola. At the vertex, an upward parabola stops going down and starts going up whereas a downward parabola stops going up and starts going down. For an upward parabola, vertex is the **lowest point** on the graph. For a downward parabola, vertex is the **highest point** on the graph. A parabola, upward or downward is symmetrical about the vertical line through the vertex. The vertical line is called the axis of the parabola.

To Draw the Graph of a Quadratic Function $f(x) = ax^2 + bx + 4$

Let the quadratic function be

$$f(x) = ax^2 + bx + c$$

or $$y = ax^2 + bx + c$$

where $a, b, c \in R, \quad a \neq 0.$

Steps for drawing a graph of a quadratic function

(*i*) Decide the shape of the parabola by checking the sign of 'a'. Upwards if $a > 0$ and Downward if $a < 0$.

(*ii*) Find $-\frac{b}{2a}$. This is the x-coordinate of vertex. The y-coordinate can be obtained by putting $x = -\frac{b}{2a}$ in equation (1).

(*iii*) Prepare a table of values for the quadratic function. Write at least 2 values of x less than $-\frac{b}{2a}$ and at least 2 values of x greater than $-\frac{b}{2a}$ with $x = -\frac{b}{2a}$ in the centre. The value $x = 0$ gives the point where the graph intersects y-axis.

(*iv*) Plot the ordered pairs given by the table of values. Pass a parabola through these points by free hand.

(*v*) Draw a vertical line through the vertex. This is the **axis of parabola**. If the graph sheet is folded along this line, then the left and right halves of the graph coincide.

To read the value of the function for a given value of *x*

[To find the value of f at x = a, i.e., to find f(a)]

On x-axis, mark the given value of x by a point A. Through A, draw a vertical line to intersect the graph at B. Through B, draw a horizontal line to meet the y-axis at C. The y coordinate of C is the value of f at $x = a$.

To read the values of *x* for a given value of *f(x)*

[To find x when f(x) = d, i.e., y = d]

On the y-axis, mark the given value of y by a point A. Through A, draw a horizontal line to intersect the graph at B *and* C. Through B and **C**, draw a vertical lines to meet the x-axis at D and E respectively. The x coordinate of D and E are the required values of x such that $f(x) = d$.

Example 11: *Draw the graph of the quadratic function* $f(x) = x^2 - 4x + 3$*.*

From the graph, find $f(x)$ *when* $x = -1$ *and when* $x = 5$*.*

Solution:

Here, $y = f(x) = x^2 - 4x + 3$ ...(1)

where $a = 1,\ b = -4,\ c = 3$

Since 'a' is positive, the graph of quadratic function is an upward parabola.

$$-\frac{b}{2a} = -\frac{-4}{2\times 1} = \frac{4}{2} = 2.$$

This is the x-coordinate of vertex.

Putting $x = 2$ in (1) $y = 2^2 - 4(2) + 3 = 4 - 8 + 3 = -1$

$\Rightarrow$ (2, –1) is the vertical of parabola.

Where $x = -11$

$y = (-1)^2 - 4(-1) + 3$

$= 1 + 4 + 3 = 8$

When $x = 0$, $y = 0 - 0 + 3 = 3$

When $x = 1$, $y = 1^2 - 4 \times 1 + 3 = 0$

When $x = 2$, $y = 2^2 - 4 \times 2 + 3 = -1$

$\Rightarrow$ Vertex is (2, –1)

When $x = 3$, $y = 3^2 - 4 \times 3 + 3 = 0$

When $x = 4$, $y = 4^2 - 4 \times 4 + 3 = 3$

When $x = 5$, $y = 5^2 - 4 \times 5 + 3 = 5$

Table of Values

x	0	1	**2**	3	4	5
y	3	0	**–1**	0	3	8
(x, y)	(0, 3)	(1, 0)	(2, –1)	(3, 0)	(4, 3)	(5, 8)

Plot the ordered pairs given by the table of values and pass a parabola through them. Draw a vertical line through the vertex. This is the axis of parabola.

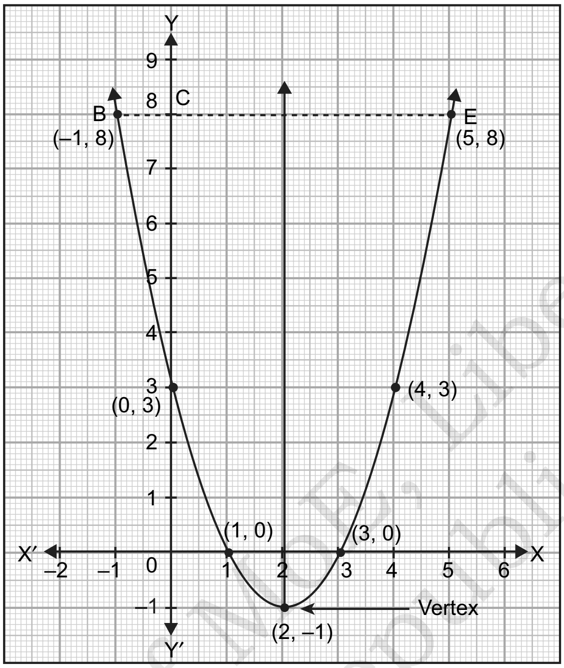

Fig. 5.2.

To find the value of f at $x = -1$

On x-axis, mark $x = -1$ by A. Through A, draw a vertical line to intersect the graph at B. Through B, draw a horizontal line to meet y-axis at C. The coordinate of C is 8.

Therefore, $f(-1) = 8$.

On x-axis, mark $x = 5$ by D. Through D, draw a vertical line to intersect the graph at E. Through E, draw a horizontal line to meet y-axis at C. The y coordinate of C is 8. Therefore, $f(5) = 8$.

Example 12: *Draw the graph of the quadratic function* $y = -2x^2 + 12x - 10$. *From the graph, find the pre-images of 6.*

Solution:

Here $\quad y = -2x^2 + 12x - 10$

where, $\quad a = -2,\ b = 12,\ c = -10$

Since 'a' is negative, the graph of quadratic function is a downward parabola.

$-\frac{b}{2a} = -\frac{12}{2\times(-2)} = -\frac{12}{-4} = 3.$ This is the x-coordinate of vertex.

$$\begin{cases} \text{when } x = 1, & y = -2\times1^2 + 12\times1 - 10 \\ & = -2 + 12 - 10 = 0 \\ \text{when } x = 1, & y = -2\times2^2 + 12\times2 - 10 \\ & = -8 + 24 - 10 = 6 \\ \text{when } x = 3, & y = -2\times3^2 + 12\times3 - 10 \\ & = -18 + 36 - 10 = 8 \end{cases}$$

$\Rightarrow$ Vertex is (3, 8)

$$\begin{cases} \text{when } x = 4, & y = -2\times4^2 + 12\times4 - 10 \\ & = -32 + 48 - 10 = 6 \\ \text{when } x = 5, & y = -2\times5^2 + 12\times5 - 10 \\ & = -50 + 60 - 10 = 0 \end{cases}$$

Table of Values

x	1	2	3	4	5
y	0	6	8	6	0
(*x*, *y*)	(1, 0)	(2, 6)	(3, 8)	(4, 6)	(5, 0)

Plot the ordered pairs given by the table of values and pass a parabola through them. Draw a vertical line through the vertex. This is the axis of parabola (Fig. 40).

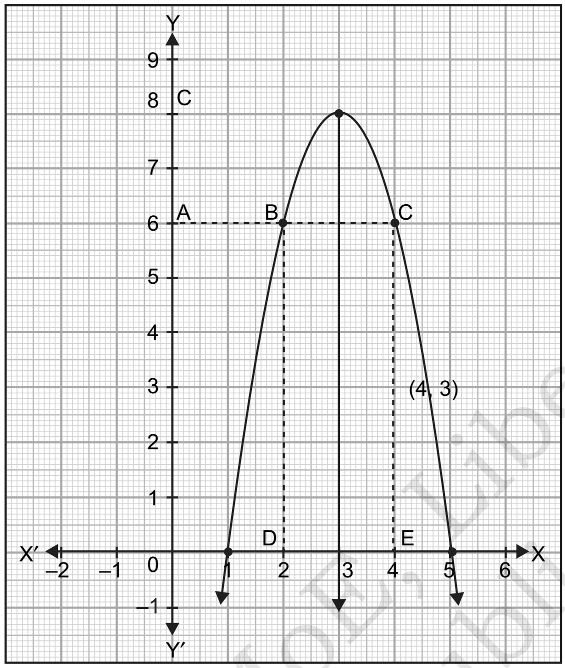

Fig. 5.3.

To find the pre-images of 6

We have to find the values of x for which $y = f(x) = 6$.

On y-axis, mark $y = 6$ by A. Through A, draw a horizontal line to intersect the graph at B and C. Through B and C, draw vertical lines to meet x-axis at D and E, respectively. The x coordinates of D and E are 2 and 4 respectively. Therefore, the desired pre-images of 6 are 2 and 4, *i.e.*, $f(2) = 6 = f(4)$.

EXERCISE 5.4

1. Draw the graph of following quadratic functions:

(i) $y = x^2 - x - 2$ for $-2 \le x \le 3$

(ii) $y = -x^2 + 2x + 3$ for $-2 \le x \le 4$

2. Draw the graph of the quadratic function $f(x) = x^2 - 3x - 4$. From the graph, find $f(x)$ when $x = -2$ and when $x = 3$.

3. Draw the graph of the quadratic function $f(x) = x^2 - 2x - 8$. From the graph, find $f(x)$ when $x = -3$ and when $x = 3$.

4. Draw the graph of the quadratic function $f(x) = -x^2 - 4x - 4$. From the graph,

(*i*) find $f(x)$ when $x = -4$. (*ii*) find pre-image of -1.

5.7. SOLVING QUADRATIC EQUATIONS

The standard form of a **quadratic equation** is:

$$ax^2 + bx + c = 0 \quad ...(1)$$

where a, b and c are real numbers nd $a \neq 0$.

If we write $f(x) = ax^2 + bx + c$ ...(2)

then (2) is a **quadratic function**.

To solve the quadratic equation (1) graphically, we draw the graph of quadratic function (2), *i.e.*, $y = ax^2 + bx + c$ where $y = f(x)$ as done in previon section. The value of x for which $y = 0$, *i.e.*, $ax^2 + bx + c = 0$ are the **solutions of equation** (1). On the graph of $y = f(x)$, roots/solutions of equation (1) are the values of x where the graph intersects x-axis or where $y = 0$.

If the graph

(i) does **not intersect *x*-axis** anywhere, then equation (1) has **no solution** or **root** or **truth value**.

The truth set $T = \phi$

(ii) **touches *x*-axis** at the point $(\alpha, 0)$, the $x = \alpha$ is the twice **repeated solution** of equation (1).

The truth set $T = \{\alpha\}$

(iii) **intersects *x*-axis** at **two distinct points** $(\alpha, 0)$ and $(\beta, 0)$, then $x = \alpha$ and $x = \beta$ are the **two solutions** of equation (1).

The truth set T = $\{\alpha, \beta\}$.

Steps for solving $ax^2 + bx + c = 0$, $a \neq 0$ graphically.

1. Let $y = ax^2 + bx + c$ ...(1)
2. Check the sign of a.

   $a > 0 \Rightarrow$ upward parabola

   $a < 0 \Rightarrow$ downward parabola

3. Find $-\dfrac{b}{2a}$. Put $x = -\dfrac{b}{2a}$ in equation (1) and find y. This gives the vertex of parabola, say (x_0, y_0).
4. Form the table of values. Give six integral values of x, three less than x_0 and three more than x_0. This gives seven ordered pairs (x, y).
5. Plot the seven ordered pairs.

6. Pass a smooth curve through all the plotted points. This is the graph of $f(x) = ax^2 + bx + c$.
7. Mark the points where the graph intersects x-axis and find the ordered pairs corresponding to them.
8. The x-coordinates of points in step (7) are the required solutions or truth values or roots of the quadratic equation $ax^2 + bx + c = 0$.

Example 13: *Draw the graph of $f(x) = x^2 - 2x + 2$ and hence solve the equation $x^2 - 2x + 2 = 0$.*

Solution:

Let $y = x^2 - 2x + 2$ ...(1)

Here $a = 1$, $b = -2$, $c = 2$

Since $a > 0$, the graph of (1) is an upward parabola.

$$-\frac{b}{2a} = -\frac{-2}{2 \times 1} = 1$$

Putting $x = 1$ in (1), we get $y = 1^2 - 2 \times 1 + 2 = 1$

$\Rightarrow$ (1, 1) is the vertex.

Table of Value for $y = x^2 - 2x + 2$

x	–2	–1	0	1	2	3	4
y	10	5	2	1	2	5	10

Plot the seven ordered pairs. Pass a smooth curve through all the plotted ordered pairs. This is the graph of $y = x^2 - 2x + 2$.

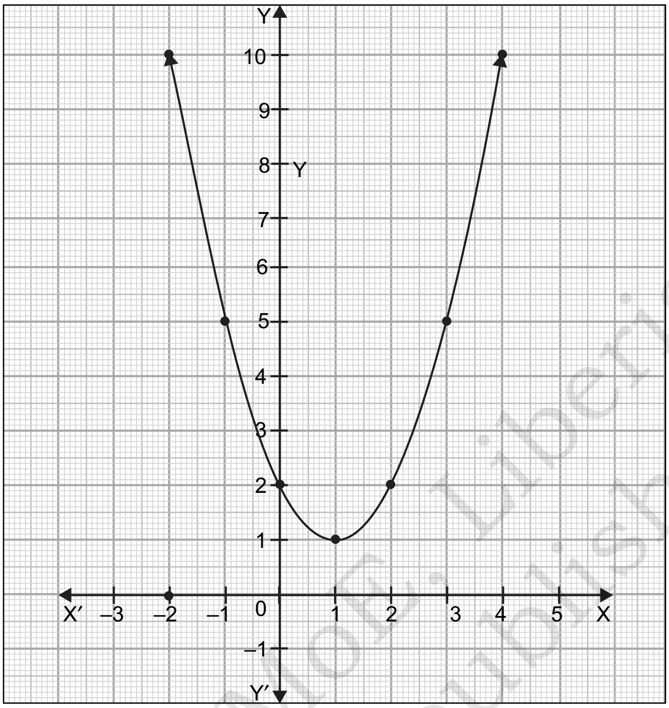

Fig. 5.4.

This is an upward parabola. The arrows at the two ends indicate that the parabola extends indefinitely in both directions. Since the graph does not intersect x-axis anywhere, the quadratic equation $x^2 - 2x + 2 = 0$ has no truth value and the truth set T = ϕ.

Example 2: *Draw the graph of $f(x) = -4x^2 - 12x - 9$ and hence solve the equation, $4x^2 - 12x - 9 = 0$.*

Solution:

Let $y = -4x^2 - 12x - 9$ ...(1)

Here $a = -4$, $b = -12$, $c = -9$

Since $a < 0$, the graph of (1) is a downward parabola.

$$-\frac{b}{2a} = \frac{-12}{2(-4)} = -\frac{3}{2}.$$

Putting $x = -\frac{3}{2}$ in (1), we get

$$y = -4\left(-\frac{3}{2}\right)^2 - 12\left(-\frac{3}{2}\right) - 9$$

$$= -4\left(-\frac{9}{4}\right)^2 + 18 - 9$$

$$= -9 + 9 = 0$$

$\Rightarrow$ $\left(-\frac{3}{2}, 0\right)$ is the vertex or turning point of (1).

Table of Values for $y = -4x^2 - 12x - 9$

x	–4	–3	–2	**–3/2**	–1	0	1
y	–25	–9	–1	0	–1	–9	–25

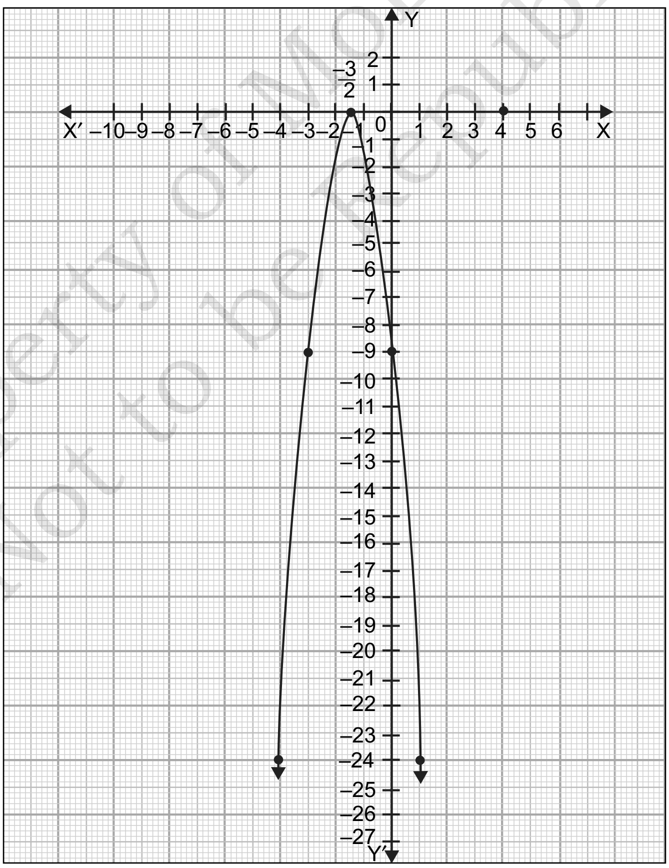

Fig. 5.5.

Plot the seven ordered pairs. Pass a smooth curve through all the plotted ordered pairs. This is the graph of $y = -4x^2 - 12x - 9$. This is a downward parabola. The arrow at the two ends indicates that the parabola extends indefinitely in both directions.

Since the graph touches x-axis at $\left(-\frac{3}{2}, 0\right)$, $x = -\frac{3}{2}$ is the only solution of the quadratic equation

$$-4x^2 - 12x - 9 = 0$$

or $$4x^2 + 12x + 9 = 0$$

Hence, the truth set T = {–3/2}

Example 3: *Solve $x^2 + 2x - 8 = 0$ graphically.*

Solution:

Let $y = x^2 + 2x - 8$ ...(1)

Here $a = 1$, $b = 2$, $c = -8$

Since $a > 0$, the graph of (1) is an unpward parabola.

$$-\frac{b}{2a} = -\frac{2}{2 \times 1} = -1$$

Putting $x = -1$ in (1), we get

$$y = (-1)^2 + 2(-1) - 8$$
$$= 1 - 10 = -9$$

$\Rightarrow$ (–1, –9) is the vertex.

Table of Values for $y = 4x^2 - 12x - 9$

x	–4	–3	–2	**–3/2**	0	1	2
y	0	–5	–8	**–9**	–8	–5	0

Plot the seven ordered pairs. Pass a smooth curve through all the plotted points. This is the graph of $y = x^2 + 2x - 8$.

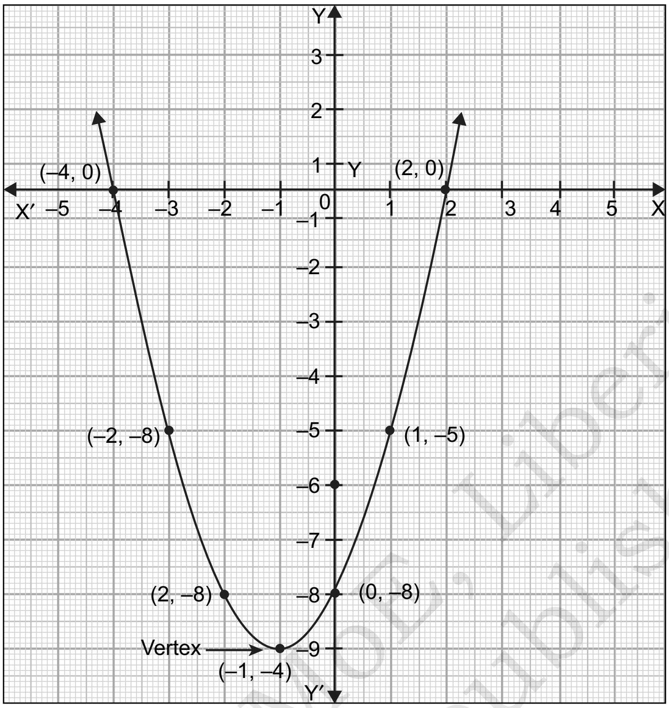

Fig. 5.6.

This is an upward parabola. The arrows at the two ends indicate that the parabola extends indefinitely in both directions. Since the graph intersects crosses x-axis at (–4, 0) and (2, 0), $x = -4$ and $x = 2$ are the two distinct solutions of the quadratic equation

$x^2 + 2x - 8 = 0$.

Hence, the truth set T = (– 4, 2)

EXERCISE 5.5

Solve the following quadratic equations graphically:

1. $x^2 - 6x + 8 = 0$
2. $x^2 - 8x + 12 = 0$
3. $-x^2 + 2x + 3 = 0$
4. $x^2 - x - 2 = 0$
5. $3 - 2x - x^2 = 0$
6. $x^2 - 6x + 9 = 0$
7. $2x^2 - 4x + 5 = 0$

MULTIPLE CHOICE QUESTIONS

1. The truth set T of the quadratic equation $3x^2 - 21x = 0$ over the set of natural number is the set

(a) $T = \{0, 7\}$ (b) $T = \{7\}$

(c) $T = \{0\}$ (d) none of these

2. The roots of the quadratic equation $x^2 - 8x + 15 = 0$ are

(a) 2, 3 (b) 3, 5 (c) 8, 15 (d) 3, –15

3. Which of the following equations has 2 as a root?

(a) $x^2 - 4x + 5 = 0$ (b) $x^2 - 3x - 12 = 0$

(c) $2x^2 - 7x + 5 = 0$ (c) $3x^2 - 6x - 2 = 0$

4. The roots of the quadratic $2x^2 - x - 6 = 0$ are

(a) $-2, \frac{3}{2}$ (b) $-2, \frac{-3}{2}$

(c) $-2, \frac{-3}{2}$ (d) $-2, \frac{3}{2}$

5. A quadratic equation $ax^2 + bx + c = 0$, $a \neq 0$ will have real and equal roots if

(a) $b^2 - 4ac > 0$ (b) $b^2 - 4ac \geq 0$

(c) $b^2 - 4ac < 0$ (d) $b^2 - 4ac = 0$

6. The quadratic equation sum of whose zeroes is $3\sqrt{2}$ and their products is 5

(a) $x^2 + 3\sqrt{2}\,x - 5 = 0$ (b) $x^2 + 5x - 3\sqrt{2} = 0$

(c) $x^2 - 3\sqrt{2}\,x + 5 = 0$ (d) $x^2 - 3\sqrt{2}\,x - 5 = 0$

7. If $p = -7$, $q = 12$ and $x^2 + px + q = 0$, then value of x is

(a) –3, – 4 (b) 3, – 4

(c) 3, 4 (d) –3, 4

8. If $x = -2$ is the root of the quadratic equation $x^2 - 3x - a = 0$ then the value of 'a' is

(a) –10 (b) 3

(c) 10 (d) –3

9. If one root of the quadratic equation $3x^2 - 11x + 6 = 0$, is 3, then, the other root is

(a) $\frac{3}{2}$ (b) 2

(c) 3 (d) $\frac{2}{3}$

10. If $\frac{1}{2}$ is a root of the equation $x^2 + kx - \frac{5}{4} = 0$, then the value of k is

(a) 2 (b) –2

(c) $\frac{1}{4}$ (d) $\frac{1}{2}$

11. The discriminant of the quadratic equation $3x^2 - 4x - 2 = 0$ is equal to

(a) 40 (b) 20

(c) 24 (d) 48

12. If $2x^2 + 12x + 18 = 0$, what is the value of x?

(a) –3 (b) –2

(c) 2 (d) 3

13. The roots or solutions of the quadratic equation whose graph is given below, are

(a) 2 and 4 (b) –4 and 2

(c) 4 and –2 (d) –2 and 2

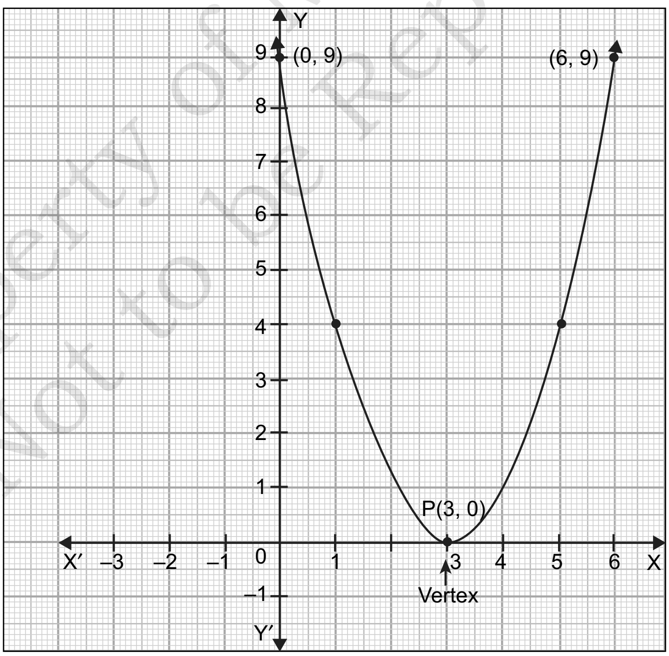

Fig. 5.7.

14. In the standard form of quadratic function $f(x) = ax^2 + bx + c$,

(a) a, b and c all are integers

(b) *a, b* and *c* all are real numbers

(c) *a, b* and *c* all are rectional numbers

(d) *a* is *a* non-zero real number and *b* and *c* are real numbers.

15. The graphic of the quadratic function $f(x) = x^2 + 1$

(a) *touches* *x*-axis at a point

(b) never intersects *y*-axis

(c) neither touches nor intersects *x*-axis

(d) intersects *x*-axis at two distinct points

16. The graph of the quadratic equation $x^2 - 4x - 5 = 0$

(a) upward parabola with turning point (2, 9)

(b) upward parabola with turning point (2, –9)

(c) downward parabola with turning point (–2, 9)

(d) downward parabola with turning point (–2, –9)